

Chaos in the Classical Analogue of the Hofstadter Problem*

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Abstract

The behaviour of an electron in a potential that resembles that of a bidimensional solid with a perpendicular magnetic field applied is studied from a classical point of view. This problem presents the standard features of chaos and some new interesting patterns. A new chaos indicator called *random walk indicator* is presented to describe some of these new patterns.

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1 Introduction

Twenty years ago D.R. Hofstadter [1] presented a celebrated work on the energy levels and wave functions of Bloch electrons on a two-dimensional lattice in rational and irrational magnetic fields, hereafter referred to as *the Hofstadter problem*. The model involves a two-dimensional square lattice of spacing a immersed in a uniform magnetic field \mathbf{B} perpendicular to it. The electrons are treated in a tight-binding approximation which amounts to considering a discrete version of the two-dimensional Laplacian in the Schrödinger equation. The resulting eigenvalue equation becomes a finite-difference equation, called Harper equation [2], whose eigenvalues can be computed by a matrix method. The magnetic flux which passes through a lattice cell, divided by a flux quantum, yields a dimensionless parameter whose rationality or irrationality highly influences the nature of the computed spectrum. When the graph of the energy spectrum is plotted over a wide range of “rational” magnetic fields, a rich recursive structure is discovered in the graph. There are large gaps whose form looks like a very striking pattern somewhat resembling a butterfly (the so called *Hofstadter butterfly*). Equally striking are the delicacy and beauty of its fine-grained structure. Likewise, the nature of the energy spectrum at “irrational” magnetic fields can also be deduced; it is shown to be a Cantor set, i.e., an uncountable but measure-zero set of points. Despite these features, it was also shown that the graph is continuous as the magnetic field varies. All these fascinating features have led us to wonder what the possible behaviours could be exhibited by a classical version of the Hofstadter problem as far as the integrability and chaotic motion is concerned.

In this work we present a classical analogue of the Hofstadter problem along with a detailed classical analysis of its integrability and onset of chaotic behaviour in several regimes. This is accomplished through the numerical study of four standard chaos indicators: Lyapunov’s exponent, power spectrum, Poincaré map and correlation function. Moreover, we present a novel chaos indicator which we call *random walk indicator* which we believe it is a suitable signal of chaos for those systems exhibiting some kind of periodicity like the lattice periodicity of our model.

The traditional way to tackle the study of a dynamic system goes through the search of constants of motion (first integrals) which reduce the dimensionality of the problem. Liouville’s Theorem states that when we have N constants of motion in involution (their Poisson brackets vanishing pairwise) the problem can be solved through quadratures. When those N constants do not exist the system usually behaves erratically and suffers from instability in the initial conditions. However, it is not easy to prove that those first integrals do not exist and in that case one has to resort to the study of some indicators which we have previously mentioned. It is admitted that there is clearly chaos when several of these indicators show this to be the case [3], [4], [5], [6].

This paper is organized as follows. In Sect. 2 we introduce a possible classical analogue of the Hofstadter model which is simple enough so as to allow a detailed analysis of its dynamics. In Sect. 3 we present the results of our numerical study of the model based on the computation of the standard indicators of chaos. In Sect. 4 we describe the onset of a new pattern of chaotic motion which strongly resembles a random walk. This leads us to introduce a new chaos indicator called *random walk indicator* to characterize this type of behaviour in systems with lattice periodicity. In Sect. 5 we study the phenomenology of the model depending on the range of the parameter space and the energy of the particle. Section 6 is devoted to conclusions and prospectives.

2 Classical Analogue of the Hofstadter Model: Equations of Motion

We shall be considering the classical motion of a charged particle, typically an electron, in a two-dimensional plane subject to the action of a static magnetic field of constant magnitude B perpendicular to the plane. We also place this charged particle under the action of a periodic potential given by,

$$V(x, y) = V_0(\cos ax + \cos ay) \quad (2.1)$$

where a is a parameter whose inverse plays the role of a *lattice spacing* in our model and V_0 sets the energy scale of the potential. This potential looks like an egg-crate when pictured as in Fig. 1. This is the way in which we model the presence of ions making up a two-dimensional crystal at a classical level. We do not claim that this model is the precise classical limit of the Hofstadter model, but nevertheless our model captures the two essential features exhibited by the quantum original model, namely, in-plane lattice periodicity plus external magnetic field applied. It is in this sense that we call it a classical analogue. Moreover, this type of periodic potential (2.1) has also been used in simple models of quantum solids, such as the Kronig-Penney model, to mimic the presence of crystal ions. Thus, the model we propose to study is suitable and simple enough to study its classical behaviour.

The Lagrangian for our classical analogue is obtained by means of the minimal coupling with the electromagnetic gauge potential \mathbf{A} . The magnetic field has components $\mathbf{B} = (0, 0, B)$ and we choose the axial gauge to describe it, namely, $\mathbf{A} = \frac{1}{2}(-Bx, By, 0)$. Thus, in Cartesian coordinates the Lagrangian reads as follows,

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tilde{V}_0(\cos ax + \cos ay) + \tilde{B}(\dot{y}x - \dot{x}y) \quad (2.2)$$

where we have set the mass of the particle $m = 1$ and introduced reduced quantities as $\tilde{V}_0 \equiv qV_0$ and $\tilde{B} \equiv qB/2$, q being the charge of the particle. The Euler-Lagrange equations of motion are,

$$\begin{cases} \ddot{x} - B\dot{y} - V_0 \sin x = 0 \\ \ddot{y} + B\dot{x} - V_0 \sin y = 0 \end{cases} \quad (2.3)$$

where we have omitted the \sim to simplify the notation henceforth and we have also set $a = 1$ so as to make $x = y = \pi$ a minimum. Equivalently, we could absorb the parameter a into a redefinition of the x, y coordinates and a further redefinition of the potential strength V_0 by $V_0 a^2$. Thus, we are left with only two independent external parameters, namely B and V_0 , characterizing our model.

The usual way to study the integrability of these equations of motion (2.3) is by searching for constants of motion according to Liouville's theorem. To achieve this in a simple fashion one seeks the underlying symmetries of the problem for then Noether's theorem guarantees us that an associated conserved quantity must exist. In our case the energy is conserved and it is given by,

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V_0(\cos ax + \cos ay) \quad (2.4)$$

We would need one more conserved quantity to render the model integrable, but as it happens, there are not further symmetries in our classical analogue model. For instance, it lacks of

central symmetry or rotational symmetry and therefore the angular momentum vector \mathbf{J} is not a conserved quantity. There is not even complete symmetry between x and y , for interchanging them would reverse the sign of \mathbf{B} , which is an axial vector. Therefore, the system has the first requisite so as to be a candidate to exhibit chaotic behaviour in some regime of the parameter space.

Before getting into the numerical study of our model it is worth noticing that there exist some approximation schemes which lead to integrable regimes. These regimes are interesting in themselves as well as providing us with some guidance when we set off chaotic regimes in next section.

A first simple approximation is achieved if we set the electric potential V_0 to zero, that is, we remove the solid picture of the model. In this case the equation of motion simplify to,

$$\begin{cases} \ddot{x} - B\dot{y} = 0 \\ \ddot{y} + B\dot{x} = 0 \end{cases} \quad (2.5)$$

and the system has both translational and central symmetry. The trajectories are simply circles. A second approximation is obtained by removing the magnetic field, thus

$$\begin{cases} \ddot{x} - V_0 \sin x = 0 \\ \ddot{y} - V_0 \sin y = 0 \end{cases} \quad (2.6)$$

we arrive at the equations for two uncoupled pendulums. The trajectories are now deformed Lissajous curves (see Fig. 2).

Lastly, another approximation can be made while retaining both the electric and magnetic fields. It amounts to linearizing the original equation of motion (2.3) as,

$$\begin{cases} \ddot{x} - B\dot{y} - V_0 x = 0 \\ \ddot{y} + B\dot{x} - V_0 y = 0 \end{cases} \quad (2.7)$$

which turn out to be exactly solvable. The trajectories look like a flower (see Fig. 3) if the electric field is predominant $V_0 \gg B$ or like “revolving loop” (see Fig. 4) if the magnetic field dominates $B \gg V_0$.

The minimal coupling also allow us to write down the Hamiltonian of the system in terms of the canonical momenta as $H = \frac{1}{2}[(p_x + By)^2 + (p_y - Bx)^2] + V_0(\cos x + \cos y)$ and we could also perform a study of the associated Hamilton-Jacobi equation, but as the system is not integrable it would not be very helpful either.

3 Standard Chaos Indicators: Numerical Calculations

From the analysis of the previous section we have realized that the search for analytical solutions of (2.3) is hopeless and the most convenient way to proceed further in the study of our classical analogue model is by means of numerical calculations.

Prior to considering the computation of standard chaos indicators, we have solved the equations of motion numerically for many different values of the external parameters V_0 and B and of the initial conditions to make sure that our numerical methods performed well. We have employed a standard numerical scheme such as the fourth-order Runge-Kutta coupled with a predictor-corrector method such as the Adams-Moulton method when it was necessary [7]. We made sure that both the rounding error and the natural error (for numerical integration of Ordinary

Differential Equations) were kept under control by two different techniques: the systematic checking of energy conservation and the checking of “inverse orbit integration”, i.e., one must get the initial conditions one departed from. In all these regards our model presents no problems.

Once we obtain the trajectory for a particular case of parameters and initial conditions, we study the following chaos indicators which we briefly explain. For an excellent introduction to standard chaos indicators the reader is referred to [8] and also [9], [10], [11] where it is standard to use several indicators to test the presence of chaotic behaviour in a classical system. The detailed analysis of these data is left for the next sections.

3.1 Poincaré Map

This indicator constitutes a qualitative visualization of the topology of a classical dynamical system in the phase space. With its help it is possible to visualize the torii associated to action-angle variables when they exist and their deformation and destruction due to non-integrable perturbations of the Hamiltonian according to the precise description given by the KAM (Kolmogorov, Arnold, Moser) theorem [12].

The Poincaré map is constructed as follows in our case. As our model has two degrees of freedom its associated phase space is four-dimensional. Let us parametrize it with coordinates (x, \dot{x}, y, \dot{y}) . In fact, we do not need all that information for we also know that the motion takes place in the energy shell domain in this phase space due to energy conservation (2.4). Thus, we may get rid of one coordinate, say y . Next we may fix the value of \dot{y} (with e.g. $\dot{y} > 0$) and hence study the phase motion in the plane (called Poincaré surface) defined by the remaining two coordinates (x, \dot{x}) . The set of points resulting from the intersection of a phase trajectory constitutes the Poincaré map. Recall that each phase trajectory is defined by giving a value of the energy E in (2.4) as well as for the model parameters V_0, B . For regular or integrable systems the Poincaré map is a well defined curve for that means that another constant of motion would exist providing another constraint to eliminate the \dot{y} coordinate in the same fashion as we did for y . Example of this kind of regular behaviour in our classical analogue model can be found in Figs. 2, 5. On the contrary, when the points of the Poincaré map fill some portion of the Poincaré surface or are randomly scattered, this is an indication of chaotic motion in the system (absence of enough first integrals to make the system integrable). We have also found this kind of chaotic behaviour in our model as can be seen in Figs. 6 a)-d) for different values of the external parameters and the energy.

3.2 Lyapunov's Exponent

One of the defining features of non-regular or chaotic motion is that the phase space gets expanded producing the separation of trajectories and at the same time, the phase space folds into itself making the trajectories to mix and blend among themselves. The Lyapunov's exponent is a measure of the first effect (expansion), or equivalently, the sensitive dependence on the initial conditions. Namely, the Lyapunov's exponent $\lambda > 0$ is computed preparing two trajectories in the phase space which are very close initially. Then, we let them evolve a number of n time steps and compute again their separation. If their separation grows exponentially as $e^{n\lambda}$, this is a possible indication of chaotic motion provided the region of the phase space is bounded like is the case for energy-conserving systems. This condition is to guarantee that the expansion effect is accompanied of the folding effect which altogether characterize the chaotic behaviour. For

instance, a simple dynamical system such as $\dot{x} = x$ also exhibits exponential separation while being regular, but nevertheless it occurs in an unbounded region.

On the contrary, for regular bounded dynamical systems the separation of nearby trajectories grows algebraically with time as t^n .

In Fig. 7 a) we plot our results for a regular behaviour while in Fig. 8 a) we show the case of chaotic motion. We plot the evolution with time for the logarithm of the separation of two initially very near trajectories divided by the time interval. If it is high (positive) during an appreciable lapse of time, the trajectories may have separated exponentially. This in turn would mean that we shall have a high degree of unpredictability in our problem for we do not have infinite accuracy to determine the initial conditions in practice.

3.3 Correlation Function

Another characteristic feature of a chaotic trajectory is the absence of any recognizable pattern in its temporal evolution. Put in another way, it embodies more information than a regular trajectory, in the same way as the decimal expansion of $1/7$ has less information than π . The correlation function is a measure of whether a dynamical system retains the memory of its past history. To this purpose, we discretize one of the coordinates, say x , and define the correlation function C_m for a m -step time interval as,

$$C_m = \frac{1}{n} \sum_{j=1}^n x'_j x'_{j+m} \quad (3.1)$$

where $x'_j \equiv x_j - \langle x \rangle = x'_j - \sum_k x_k / n$. Thus, the correlation function indicates how similar is x' (deviation of the coordinate x from its time average) to its value m -time intervals after. Therefore, a rapidly decreasing correlation function C_m is an indication of chaos, while if it remains finite it is an indication of regular motion.

In Fig. 7 b) we plot our results for a regular behaviour while in Fig. 8 b) we show the case of chaotic motion.

3.4 Power Spectrum

The most useful and appropriate way to characterize a periodic (regular) motion is through the analysis of its Fourier spectrum of frequencies. Periodicity is translated into the appearance of a peak at the characteristic frequency of the dynamical system. Quasiperiodicity amounts to a finite number of peaks corresponding to a set of principal frequencies and their harmonics. However, when we find a continuum spectrum we can suspect that the motion occurring is very complex.

The power spectrum is defined as the intensity of the Fourier transform for one discretized coordinate, say x_j . Namely, $E_k \equiv |\hat{x}_k|^2$. According to Wiener-Khinchin's theorem [8], the correlation function and the power spectrum are related through a Fourier transform.

In Fig. 7 c) we plot our results for a regular behaviour while in Fig. 8 c) we show the case of chaotic motion.

4 A New Indicator: The Random Walk Indicator

The analysis of the standard chaos indicators in the previous section shows us that our classical analogue model is rich enough so as to exhibit both regular and chaotic behaviour depending on the range of the parameter space V_0, B and on the value of the energy of the system. Furthermore, we shall show that our model also exhibits another type of behaviour which we call random walk or brownian motion whose origin can be traced back to the existence of a lattice periodicity in our system. This is the new ingredient which allows us to introduce a new chaos indicator which we believe it is a suitable one for systems presenting lattice periodicity of some sort.

Likewise the harmonic oscillator is the prototype of periodic system, a random walk is one of the most aleatory dynamical systems when can think of. A simple way to construct a random walk in the plane as shown in Fig. 9 a) is by leaving a particle to hop in a two-dimensional lattice so that at each time the particle chooses wheather to jump to the left-right or up-down by tossing two coins. Remarkably enough, we have found a region in the parameter space of our classical analogue model where the open trajectory underwent by the particle clearly looks like to that of a brownian motion as we show in Fig. 9 b).

Likewise the departure from the oscillator behaviour is a sign of chaos which is detected by the power spectrum, the proximity to a random walk can be considered a sign of chaos as well. We have come up with a way to characterize this proximity by means of the random walk indicator. Thus, if the steps jumped by a particle undergoing a random walk are of equal length, say one lattice spacing, after N steps the most probable position of the particle will be the initial position. However, the mean square value of the position is just N . Let $\mathbf{r}(t)$ be the position vector of the particle in Fig. 9 b) at the time t elapsed by hopping from the origin to a certain point of the total walk. Let us denote by $\langle \mathbf{r}^2 \rangle$ the time average over the whole random walk in Fig. 9 b), namely, $\langle \mathbf{r}^2 \rangle = \frac{1}{T} \int_0^T \mathbf{r}^2 dt$, $T \equiv N\Delta t$. We may say that a random walk has a “critical exponent” $1/2$, because r.m.s. distance grows as the square root of the elapsed time. Therefore, we may introduce an indicator denoted by I_{RW} as the quotient of the mean square value $\langle \mathbf{r}^2 \rangle$ to the number of total steps, i.e.,

$$I_{RW} \equiv \frac{\langle \mathbf{r}^2 \rangle}{T} \quad (4.1)$$

Therefore, for a random walk I_{RW} tends to a constant value equal to 1 as $N \rightarrow \infty$.

To make the checking implicit in the random walk indicator it is required that the the time elapsed be long enough so that the particle has left the unit cell of the lattice and the space under observation must be also large enough so as to comprise several unit cells for the hopping motion of the particle to make sense. This is how the necessity for lattice periodicity comes about. Now, when I_{RW} tends to a constant value, as shown in Fig. 10 a) for a real random walk, we may consider it as an indication of chaotic behaviour of this kind. In this fashion, the plot of I_{RW} for the trajectory depicted in Fig. 9 b) is remarkably similar to that of the real random walk. To reinforce this point of view, we have also plotted this indicator for the regular motion of Fig. 7 showing that $I_{RW} \rightarrow 0$ as the characteristic feature of bounded motions in our scheme.

5 Analysis of Results: Phenomenology of the System

In our search for non-usual behaviours in a classical system such as our classical analogue model, we have found in the previous section the remarkable onset of brownian motion when appropriate

large time and space scales are considered for a certain range of the parameter space (V_0, B) and energy E . We have also found standard regular motions and a variety of chaotic behaviour which we hereby describe in some more detail. We do this analysis depending on the value of the energy chosen. Our favorite initial condition to start with has been launching the charged particle from a minimum of the egg-crate potential (2.1) towards a maximum with a velocity given by the specified energy.

5.1 Negative or Low Energies

In this case the particle has not enough energy to scape and go to visit another unit cell of the periodic potential (2.1). Thus, the particle traces a sort of loops inside one cell of a more or less complicated shape. We have found all sorts of them. From time to time, a “peak” in the trajectory (sudden change of direction) occurs. This happens when the particle approaches a saddle point of the potential.

We have found chaos whenever such “peaks” show up. It comes out as a sort of “line broadening” of the curve forming the Poincaré map. This is the first step towards the filling of a whole region signalling chaos. For a given B there is a minimum value for V_0 below which there is no chaos, and there is also an upper value for V_0 above which there is regularity ever after. There are slow-motion regular trajectories which are fine, but there also exist rapid-oscillatory motions which we call “soft chaos”, for they present high Lyapunov exponent, but a correct (regular) Poincaré map and power spectrum.

We have collected all our qualitative data in the schematic Table 1 showing the qualitative behaviours found for $E = 0$ (Low Energy regime) and several values of the external parameters (V_0, B) . Maps for lower energies are alike, but there is no chaos and there is only the previously mentioned broadening of lines in the Poincaré maps. Another interesting feature is the appearance of a “non-uniform space filling” of the Poincaré surface as depicted in Figs. 6 a) and b). Thus, chaos is present as usual when a full area is filled.

5.2 High Energies

For these values of the energy the trajectory is bounded if the magnetic field is high enough, that is to say, when the intensity of the cyclotron frequency prevails in the Fourier spectrum. As a consequence, the motion is close to regular.

Another feature for this energy regime is that the trajectory has always a sensitive dependence on the initial conditions for the electron crosses several cells of the periodic potential, so it can pass close to unstable points, such as the maxima of the egg-crate potential (2.1) (if energies were lower, those points would not be reached). The Lyapunov exponent is then high and steadily growing. In this case the Poincaré map as it has been defined in Sect. 3 a) is a bad illustration because it is difficult to discern between line-broadening case and a space-filling case due to the fact that the particle travels many unit cells. Thus, we have chosen to work in a reduced Poincaré surface consisting of just one unit cell due to the periodicity of our model (i.e., we apply periodic boundary conditions converting the space into a torus). This makes the problem of drawing an easier one for now the Poincaré map is bounded. Otherwise it covers an indefinite region in the phase space and it is difficult to classify the different patterns of behaviour. In summary, we have found the following patterns:

- *Chaotic motion:* it appears very early when the particle has enough energy to visit several cells. Regular trajectories are associated with a very low or very high value for any of the parameters V_0 or B (there must be hegemony of one parameter upon the other in order to have regularity).
- *Temporally bounded trajectories:* a particle orbits around a minimum of the potential for say 18 loops and when one thinks it has been captured, it scapes and wanders erratically around the lattice. Eventually, it behaves like a random walk as in Fig. 9 b).
- *Mixed Complex Behaviour:* Here almost all chaos signals “light on” simultaneously: Poincaré maps with space filling, high Lyapunov exponents and random walks. Nevertheless, the correlation function keeps high values for the particle behaves in almost the same fashion in every cell it visits, with only some peaks, loops or sudden changes in direction from time to time.

Likewise, we illustrate all our qualitative data in the schematic Table 2 showing the qualitative behaviours found for $E = 12$ (High Energy regime) and several values of the external parameters (V_0, B) .

6 Conclusions

We have presented in this paper a possible classical analogue of the Hofstadter model [1] described in the introduction along with a detailed study of a great variety of classical dynamical behaviours exhibited by our simple model. Some of the patterns found for the Poincaré maps described in the last section we believe are specific of a system with spatial lattice periodicity such as ours. We have also employed other standard chaos indicators to carry out this analysis. Moreover, the finding, in a certain range of the parameter space (V_0, B) of a pattern of behaviour which strongly resembles a random walk or brownian motion, see Fig. 9 b), has led us to introduce an additional chaos indicator to check the onset of a random walk behaviour in systems with some sort of lattice spacing.

In this present work we have been clearly working at a very descriptive level and we believe that a more thorough understanding of the intricacies of these types of models is necessary. For instance, the connection between the classical and quantum versions of the Hofstadter model deserves more attention regarding the relation between chaos and quantization of Hamiltonian systems which is an open research field [13].

All these considerations make us believe that the study of classical dynamical systems with some sort of lattice periodicity might quite well embody new features not present in standard studies of chaotic behaviour.

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Table captions

Table 1 : Schematic table showing the qualitative behaviours found for $E = 0$ (Low Energy regime) and several values of the external parameters (V_0, B) . The meaning of the symbols are: \bullet = Space-filling in the Poincaré map, \square = Non-Uniform space filling, \parallel = Line-Broadening. P = Quasiperiodic motion.

Table 2 : Schematic table showing the qualitative behaviours found for $E = 12$ (High Energy regime) and several values of the external parameters (V_0, B) (the meaning of the symbols are as in table 1).

Figure captions

Figure 1 : Picture of the periodic potential in Eq. (2.1) employed to mimick a crystal solid in our classical analogue of the Hofstadter model.

Figure 2 : A typical trajectory in configuration space for $B = 0$ along with its Poincaré map.

Figure 3 : Flower-like trajectory for the linear approximation when $V_0 \gg B$.

Figure 4 : Revolving-loop trajectory for the linear approximation when $B \gg V_0$.

Figure 5 : Poincaré map for $V_0 = 3, B = 1, E = 0$.

Figure 6 : a) Poincaré map for $V_0 = 16, B = 5, E = 2$. b) Poincaré map for $V_0 = 7, B = 4, E = 0$. c) Poincaré map for $V_0 = 4, B = 3, E = 12$. d) Poincaré map for $V_0 = 4, B = 6, E = 12$.

Figure 7 : Plots of some standard chaos indicators for $V_0 = 3, B = 1, E = 0$. a) Lyapunov exponent. b) Correlation function. c) Power spectrum.

Figure 8 : Plots of some standard chaos indicators for $V_0 = 7, B = 4, E = 0$. a) Lyapunov exponent. b) Correlation function. c) Power spectrum.

Figure 9 : a) Picture of a real two-dimensional random walk. b) Plot of the random walk appearing in our classical analogue model for $V_0 = 16, B = 5, E = 2$.

Figure 10 : Plot of the random walk indicator I_{RW} for: a) a real two-dimensional random walk. b) the random walk appearing in our classical analogue model for $V_0 = 16, B = 5, E = 2$. c) a regular motion appearing for $V_0 = 3, B = 1, E = 0$.

$B V_0$	1	2	2.05	2.5	3	4	4.5	5	6	7	8	9	10	11	12
2	P	P	P	P	P	P	P	P	P	P	P	P	P	P	P
2.5	P	P		•	•	P	P	P	P	P	P	P	P	P	P
3	P	P	P	P	•	•		P	P	P	P	P	P	P	P
4	P	P	P	P	P	P	•	•	•	□		P	P	P	P
5	P	P	P	P	P	P	P	P	P	P		•	•	•	•
6	P	P	P	P	P	P	P	P	P	P	P	P	P		•

Table 1: Schematic table showing the qualitative behaviours found for $E = 0$ (Low Energy regime) and several values of the external parameters (V_0, B) . The meaning of the symbols are: • = Space-filling in the Poincaré map, □ = Non-Uniform space filling, || = Line-Broadening. P = Quasiperiodic motion.

$B V_0$	1	2	4	5	6	7	8	9	10	11	12
0.1	□	P	P	P	P	P	P	P	P	P	P
0.5	•	•	•	•	•	•	•	•	•	•	•
1			•	•	•	•	P	•	•	•	•
2	P	P	•	•	•	•	•	P	P	P	P
3	P	P	P	•	P	P	P	P	P	P	P
4	P	P	P	P	P	P	P	P	P	P	P
5	P	P	P	P	P	P	P	P	P	P	P

Table 2: Schematic table showing the qualitative behaviours found for $E = 12$ (High Energy regime) and several values of the external parameters (V_0, B) (the meaning of the symbols are as in table 1).

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